Binomial Probabilities from Binomial Theorem

1. Introduction

a) Binomial probabilities

Binomial probabilities relate to situations in which there are only *two* possible outcomes, classically called ‘success’ and ‘failure’, for each attempt/run/trial. An example of this might be scoring a ‘bullseye’ in darts, or getting a ‘strike’ in bowling. When only one attempt is made, the probability of success is p (*e.g.* 0.20 = 20 %), while the probability of failure is q (*e.g.* 0.80 = 80 %); notice that $p + q = 1$.

When multiple attempts are made, several different overall outcomes are possible, depending upon the outcomes at each attempt. For example, if two attempts are made, this might result in a total of two successes, or two failures, or one of each. The likelihood of obtaining a certain overall outcome can be computed based upon the number of attempts (n), the number of successes in that outcome (x), and the probability p, namely ${}^{n}C_{x}\, p^{x}\, (1 - p)^{n-x}$, in which C represents the ‘combinations’ function. This can be compactly written ${}^{n}C_{x}\, p^{x}\, q^{n-x}$.

Note carefully that the minimum number of successes is *zero* (x=0), but by convention this provides the *first* term in the sequence. Thus, ${}^{n}C_{x}\, p^{x}\, q^{n-x}$ represents the $\{x+1\}^{\text{th}}$ term (T_{x+1}).

Occasionally it is preferred to relate the terms to a simpler ordinal variable; if we want to write down the r^{th} term (T_r), it will be ${}^{n}C_{r-1}\, p^{r-1}\, q^{n-r+1}$. This is consistent, as clearly $r = x+1$.

b) Binomial theorem

The **binomial theorem** relates to expansion of an expression with *two* terms that has been raised to some positive-integer power. For example, $(1 + z)^9$, or $(A - B)^N$. The terms in these expansions are well defined and all follow the same pattern.

For $(q + p)^n$ they are:

$$ {}^{n}C_{0}\, p^{0}\, q^{n} + {}^{n}C_{1}\, p^{1}\, q^{n-1} + {}^{n}C_{2}\, p^{2}\, q^{n-2} + \ldots + {}^{n}C_{x}\, p^{x}\, q^{n-x} + \ldots + {}^{n}C_{n-1}\, p^{n-1}\, q^{1} + {}^{n}C_{n}\, p^{n}\, q^{0}, $$

or equivalently

$$ {}^{n}C_{0}\, p^{0}\, q^{n} + {}^{n}C_{1}\, p^{1}\, q^{n-1} + {}^{n}C_{2}\, p^{2}\, q^{n-2} + \ldots + {}^{n}C_{r-1}\, p^{r-1}\, q^{n-r+1} + \ldots + {}^{n}C_{n-1}\, p^{n-1}\, q^{1} + {}^{n}C_{n}\, p^{n}\, q^{0}, $$

in which T_{x+1} [or T_r] is the $\{x+1\}^{\text{th}}$ [or r^{th}] term in the expansion — starting with the *first* term. Notice that these terms are each the same as the terms needed to compute the probabilities of specific overall outcomes in problems relating to binomial probability.

2. The overall outcome with the highest probability

In most problems the probability per attempt (p), the number of successes (x) and the number of attempts (n) are known, and the probability of that overall outcome is unknown.

However, sometimes it is of interest to consider which overall outcome would have the highest probability, for a given probability per attempt (p) and number of attempts (n).

Note that this corresponds to finding the mode of a binomial probability distribution.

There are four ways to find the overall outcome that has the highest probability, as summarised in the following table.

Method	Complexity	Computational effort	Usefulness in proofs
Brute force	Low	High	Medium
Targetting based on mean	Low	Low	Medium
Split-function algorithm	Medium	Low	Low
Ratio of probabilities	High	Low	High

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In the N.S.W. Year 12 HSC syllabus it is generally *anticipated* that the method involving ratios of consecutive probabilities will be used.

Sample problem

If the likelihood that a tennis player wins a point is equal to 0.45 (p=0.45), then if four points have been played (n=4), what is the outcome with the highest probability?

a) Brute force approach

The simplest approach involves calculating the probabilities for *all* of the possible outcomes.

The possible outcomes are distinguished as combinations, in which the order that the points were won or lost is irrelevant.

Effectively the greater or smaller number of permutations in which a certain combination could exist are accounted for by the coefficient ${}^nC_x$. [or ${}^nC_{r-1}$].

For the problem as stated, the number of possible outcomes is five: 0, 1, 2, 3, 4, or 5 points successfully won. Notice that this is equal to n+1, because of the need to account for the scenario of zero successes.

Calculation according to the above formulæ yields:

Number of successes	Probability of this overall outcome
0	0.09150625
1	0.299475
2	0.3675375
3	0.200475
4	0.04100625

It is important to observe that the sum of all these individual probabilities is equal to 1 (*i.e.* 100 %). This must always be true, if all possible scenarios are included.

It can also be seen by recognising that the sum is identically equal to the binomial expansion of $(q + p)^n$, and knowing that $p + q = 1$, this is simply $(1)^n = 1$.

By inspection of the table it is obvious at a glance that the outcome with the highest probability is 2 successes (*i.e.* tennis points won).

This scenario could be achieved in many ways, namely SSLL, SLSL, SLLS, LSSL, LSLS, LLSS, which are all considerer equivalent in this analysis. ("S" represents a success, and "L" represents a loss.)

b) Targetting based on the 'expected value' (*i.e.* the mean)

For small values of n, the naïve brute-force approach is quickest and easiest to apply. However, when n isn't small (*e.g.* greater than 4), it is expedient to reduce the number of scenarios to consider by first computing the 'expected value'.

For these problems the 'expected value' will be the mean of the binomial distribution, and it is very easily and quickly calculated as simply $n\,p$.

While this may not immediately appear familiar, it tends to match the intuition that we have developed. For example, if the chance that an engineer makes a new invention in a given year is 40 %, then over a ten-year period we *expect* that she would — 'on average' — make 4 new inventions, which is equal to 0.40 × 10.

But note that if the chance that her colleague makes a new invention in a given year is 35 %, then over a ten-year period we *expect* that he would — 'on average' — make 3½ new inventions, which is equal to 0.35 × 10. Given that there is no such thing as half an invention, practically we interpret this as meaning 'about 3 or 4'.

Although there seems to be little or no publicising of the fact, it turns out that the **mean** of the binomial distribution is a pretty good approximation for the **mode** of the binomial distribution. However, the mean is much simpler to calculate.

WARNING: For many other probability distributions the mean and the mode are very different from one another!

For the problem as stated, the mean must be 0.45×4 = 1.8 points won. Notice that this is not an integer, which is typical of mean values. This tells us that the mode might be 1 success or

2 successes within the 4 points played. It is important to note that there is no obvious way to determine in advance whether we need to round up or round down from the mean to obtain the mode.

First we would calculate the probabilities for these candidate solutions.

Number of successes	Probability of this overall outcome
1	0.299475
2	0.3675375

These two quick calculations demonstrate that 1 success cannot be the scenario with the highest probability. It looks like the correct solution may be 2 successes. However, for a rigorous proof we need to confirm this by checking that it really is a maximum, *i.e.* that both of the neighbouring probability terms are smaller. This involves just one more calculation.

Number of successes	Probability of this overall outcome
1	0.299475
2	0.3675375
3	0.200475

This might seem to be a 'proof', but to be completely rigorous we need to recall also one property of the binomial distribution, which is that it is 'unimodal' — there will be only one peak. As we have clearly identified the peak, this must be the solution: the outcome with the highest probability is 2 successes (*i.e.* tennis points won).

Other probability distributions, besides the binomial distribution, could have multiple peaks, so just examining the neighbourhood of one peak would not be sufficient to prove that there was not another peak corresponding to scenarios with even higher probabilities.

c) Applying a split-function algorithm*

Usually the mode of a binomial distribution, $B(n,p)$, is given by $\lfloor (n+1)p \rfloor = \lfloor n\,p + p \rfloor$, where $\lfloor \cdots \rfloor$ represents the 'floor' function.

Notice the similarity to the formula for the mean: $\lfloor n\,p + p \rfloor = \lfloor \text{mean} + p \rfloor$.

The 'floor' function basically means round <u>down</u>.

However, when $(n+1)p$ is an integer and p is neither 0 nor 1, then the distribution has two modes that are immediately adjacent to each other: $(n+1)p$ and $(n+1)p - 1$. When p is equal to 0 or 1, then the mode will be respectively 0 or n.

The three cases can be summarised as follows:

$$\text{mode} = \begin{cases} 0 & \text{for } p = 0 \\ \lfloor (n+1)p \rfloor & \text{when } (n+1)p \text{ is not an integer} \\ (n+1)p \text{ and } (n+1)p - 1 & \text{when } (n+1)p \in \{1, \cdots, n\} \\ n & \text{for } p = 1 \end{cases}$$

Although this can be proved to be correct†, by itself it does not constitute any sort of proof (unless the algorithm were accepted as correct at the outset).

For the problem as stated, $p \neq 0$ and $p \neq 1$, and we compute $(n+1)p = 5 \times 0.45 = 2.25$, which is not an integer, so the mode must be given by $\lfloor (n+1)p \rfloor = \lfloor 2.25 \rfloor = 2$.

Thus we find the same solution as before: the outcome with the highest probability is 2 successes (*i.e.* tennis points won).

* Adopted from https://en.wikipedia.org/wiki/Binomial_distribution#Mode

† *Ibidem.*

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d) Ratio of consecutive binomial probabilities

Considering the ratio of consecutive probabilities in the sequence/expansion is *analogous* to considering the variation in local gradient of a function. We know that to find a turning point (a local maximum or minimum) we should find where the gradient is equal to zero. The difference here is that to find the maximum (*i.e.* the mode) we should find where the ratio of consecutive probabilities is equal to one.

Expressed with respect to the number of successes, x, the ratio can in general be written:

$$\frac{T_{(x+1)+1}}{T_{x+1}} = \frac{{}^{n}C_{x+1}\, p^{x+1}\, q^{n-x-1}}{{}^{n}C_{x}\, p^{x}\, q^{n-x}} = \frac{\frac{n!}{(x+1)!\,(n-x-1)!}\, p}{\frac{n!}{x!\,(n-x)!}\, q} = \frac{x!\,(n-x)!\, p}{(x+1)!\,(n-x-1)!\, q} = \frac{(n-x)\, p}{(x+1)\, q}\ .$$

T_{x+1} and $T_{(x+1)+1}$ are respectively the $\{x+1\}^{\text{th}}$ and $\{(x+1)+1\}^{\text{th}}$ terms.

Alternatively (and equivalently), the same ratio can be written expressed with respect to r, referring to the r^{th} and $\{r+1\}^{\text{th}}$ terms, as generally:

$$\frac{T_{r+1}}{T_r} = \frac{{}^{n}C_{r}\, p^{r}\, q^{n-r}}{{}^{n}C_{r-1}\, p^{r-1}\, q^{n-r+1}} = \frac{\frac{n!}{r!\,(n-r)!}\, p}{\frac{n!}{(r-1)!\,(n-r+1)!}\, q} = \frac{(r-1)!\,(n-r+1)!\, p}{r!\,(n-r)!\, q} = \frac{(n-r+1)\, p}{r\, q}\ .$$

It is also just as valid to calculate the reciprocals of these (equivalent) ratios.

Either of these expressions can be set equal to one, and then solved for respectively x or r.

Occasionally this initial estimate of x or r will be an integer, which means that both terms (T_{x+1} and $T_{(x+1)+1}$, or T_r and T_{r+1}) are equal, so both of them represent the modes.

However, the value obtained as an initial estimate will often not be an integer. In that case, it indicates that the probability is still increasing from T_{x+1} to $T_{(x+1)+1}$ [or from T_r to T_{r+1}] — and only after that does the probability decrease. That means that the mode must be $T_{(x+1)+1} = T_{r+1}$. Therefore it is necessary to round up (never round down!).

For the problem as stated, to apply this technique we need to find out when the ratio is equal to 1.

Thus we can solve $(n-x)p = (x+1)q$. That is, $0.45(4-x) = 0.55(x+1)$. Hence, $x = 1.25$. This means that the mode must be $x = \lceil 1.25 \rceil = 2$.

$\lceil \cdots \rceil$ represents the 'ceiling' function, which basically means round up.

Equivalently, we could instead solve $(n-r+1)p = r\,q$. That is, $0.45(4-r+1) = 0.55\,r$. Hence, $r = 2.25$. This means that the mode must be $r = \lceil 2.25 \rceil = 3$. Given that $r = x+1$, the mode can also be represented by $x = r - 1 = 2$.

Thus we find the same solution as before: the outcome with the highest probability is 2 successes (*i.e.* tennis points won).

Of course, this is confirmed to be consistent with the probabilities found for each of the five possible overall outcomes.

Term number r	**Number of successes** x	**Probability of this overall outcome** $P(x)$
1	0	0.09150625
2	1	0.299475
3	2	0.3675375
4	3	0.200475
5	4	0.04100625

3. Further examples

To better illustrate the various scenarios that can arise, a variety of further examples are presented on the following pages.

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BINOMIAL PROBABILITIES
EXPLOITING THE
BINOMIAL THEOREM

Probability of outcome being x successes from n trials

= P(x)

= T_{x+1} = T_r

= ${}^nC_x\, p^x\, q^{(n-x)}$ = ${}^nC_{r-1}\, p^{r-1}\, q^{(n-r+1)}$

in which

p is the probability of success for each attempt

q is the probability of failure (*i.e.* not success) for each attempt

T_r is the r^{th} term in the series/expansion

and

$r = x + 1$ (The first term is for zero successes.)

$p + q = 1$ (Probabilities for all outomes sum to one.)

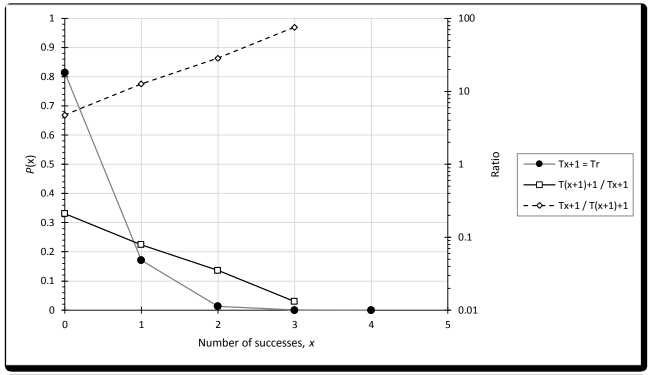
Scenario 1

n = 4
p = 0.05
∴ q = 0.95

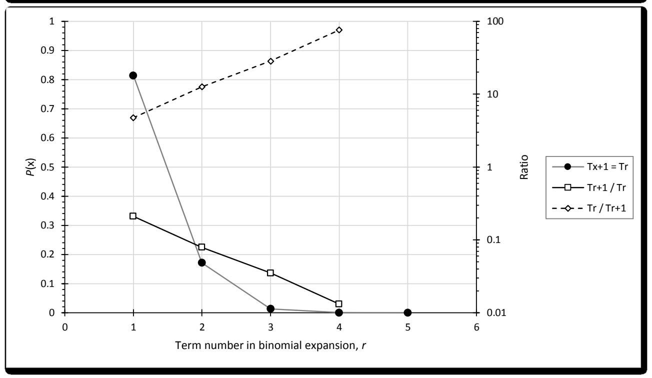
Term number	**Number of successes**	**P(x)**	**Ratios in x:**		**Ratios in r:**	
r	**x**	**$T_{x+1} = T_r$**	**$T_{(x+1)+1} / T_{x+1}$**	**$T_{x+1} / T_{(x+1)+1}$**	**T_{r+1} / T_r**	**T_r / T_{r+1}**
0	-1	0	#DIV/0!	0	#DIV/0!	0
1	**0**	**0.81450625**	0.210526316	4.75	0.210526316	4.75
2	1	0.171475	0.078947368	12.66666667	0.078947368	12.66666667
3	2	0.0135375	0.035087719	28.5	0.035087719	28.5
4	3	0.000475	0.013157895	76	0.013157895	76
5	4	0.00000625	0	#DIV/0!	0	#DIV/0!
6	5	0	#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!

Sum: 1

Graphs based on plotting data as functions of *x*. ===>
Read probability from left axis (linear scale).
Read ratios of probabilities from right axis (logarithmic scale).



Graphs based on plotting data as functions of *r*. ===>
Read probability from left axis (linear scale).
Read ratios of probabilities from right axis (logarithmic scale).



EXTRAPOLATION:
Ratios equal 1 for ...

... x = -0.75 = (n + 1) / (q/p + 1) − 1

... r = 0.25 = (n + 1) / (q/p + 1)

This rounding is predictable. ===>

Round UP to yield maximum probability for ...

... outcome, x = 0 successes.

... term, r = 1 in the series/expansion.

This rounding is <u>un</u>predictable. ===>
(However, computing an expected value and checking probabilities of the neighbouring outcomes is reliable, and arguably easier!)

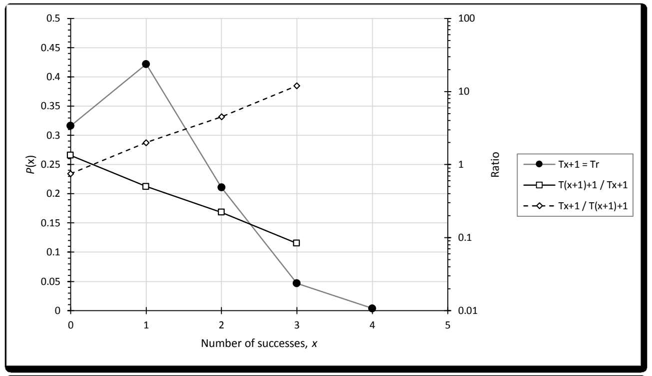
Just for comparison, the 'expected value' = $p\ n$ = 0.2 successes
This neighbours the most probable outcome(s),
which can be obtained (<u>here</u>) by rounding DOWN.

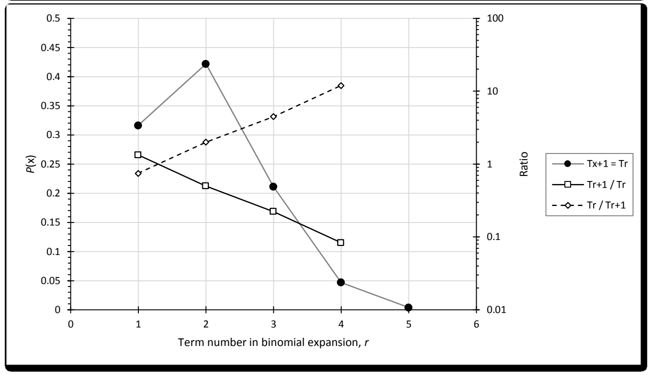
Scenario 2

n = 4
p = 0.25
∴ q = 0.75

Term number	Number of successes	P(x)	Ratios in x:		Ratios in r:	
r	x	$T_{x+1} = T_r$	$T_{(x+1)+1} / T_{x+1}$	$T_{x+1} / T_{(x+1)+1}$	T_{r+1} / T_r	T_r / T_{r+1}
0	-1	0	#DIV/0!	0	#DIV/0!	0
1	0	0.31640625	1.333333333	0.75	1.333333333	0.75
2	**1**	**0.421875**	0.5	2	0.5	2
3	2	0.2109375	0.222222222	4.5	0.222222222	4.5
4	3	0.046875	0.083333333	12	0.083333333	12
5	4	0.00390625	0	#DIV/0!	0	#DIV/0!
6	5	0	#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!

Sum: 1





INTERPOLATION:
Ratios equal 1 for ...
... x = 0.25 $= (n + 1) / (q/p + 1) - 1$
... r = 1.25 $= (n + 1) / (q/p + 1)$

Round UP to yield maximum probability for ...
... outcome, x = 1 successes.
... term, r = 2 in the series/expansion.

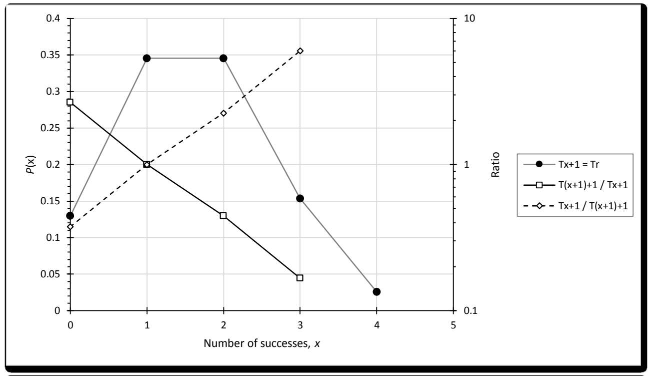
Just for comparison, the 'expected value' = $p\ n$ = 1 successes
This neighbours the most probable outcome(s),
which can be obtained (here) DIRECTLY.

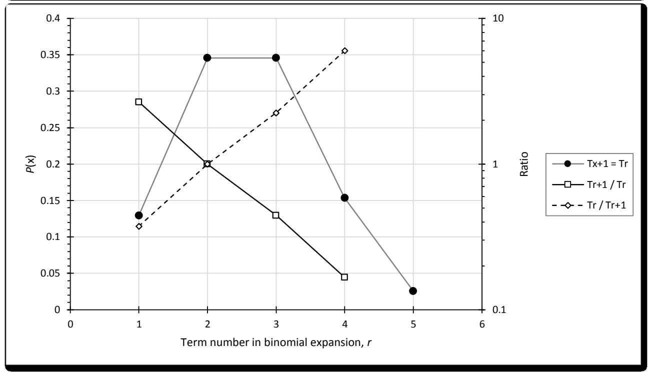
Scenario 3

n = 4
p = 0.4
∴ q = 0.6

Term number	Number of successes	P(x)	Ratios in x:		Ratios in r:	
r	x	$T_{x+1} = T_r$	$T_{(x+1)+1} / T_{x+1}$	$T_{x+1} / T_{(x+1)+1}$	T_{r+1} / T_r	T_r / T_{r+1}
0	-1	0	#DIV/0!	0	#DIV/0!	0
1	0	0.1296	2.666666667	0.375	2.666666667	0.375
2	**1**	**0.3456**	1	1	1	1
3	**2**	**0.3456**	0.444444444	2.25	0.444444444	2.25
4	3	0.1536	0.166666667	6	0.166666667	6
5	4	0.0256	0	#DIV/0!	0	#DIV/0!
6	5	0	#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!

Sum: 1





INTERPOLATION:
Ratios equal 1 for ...
... x = 1 $= (n + 1) / (q/p + 1) - 1$
... r = 2 $= (n + 1) / (q/p + 1)$

Take this value DIRECTLY, and ALSO ADD ONE for ...
... outcome, x = 1 , and ALSO 2 successes.
... term, r = 2 , and ALSO 3 in the series/expansion.

Just for comparison, the 'expected value' = $p\ n$ = 1.6 successes
This neighbours the most probable outcome(s),
which can be obtained (here) by rounding both UP and DOWN.

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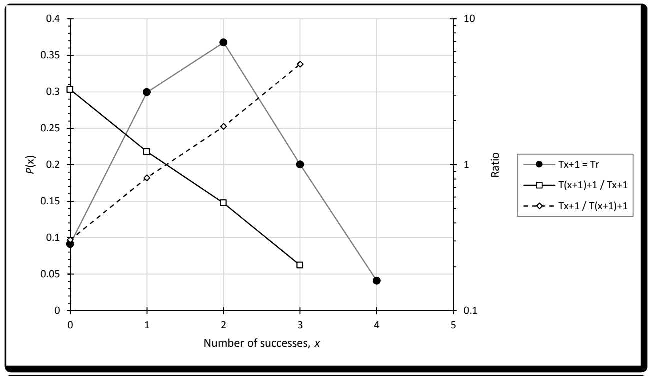
#1

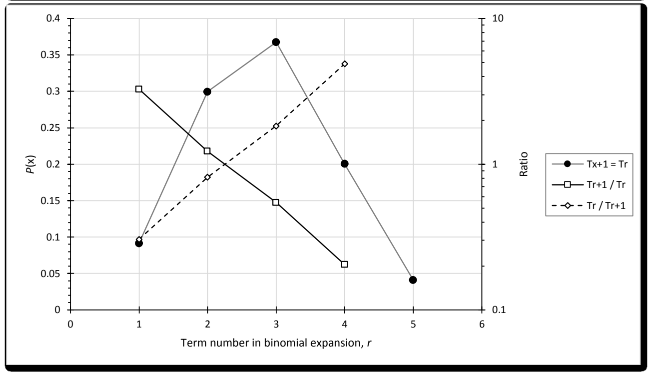
Scenario 4

n = 4
p = 0.45
∴ q = 0.55

Term number	Number of successes	P(x)	Ratios in x:		Ratios in r:	
r	x	$T_{x+1} = T_r$	$T_{(x+1)+1} / T_{x+1}$	$T_{x+1} / T_{(x+1)+1}$	T_{r+1} / T_r	T_r / T_{r+1}
0	-1	0	#DIV/0!	0	#DIV/0!	0
1	0	0.09150625	3.272727273	0.305555556	3.272727273	0.305555556
2	1	0.299475	1.227272727	0.814814815	1.227272727	0.814814815
3	**2**	**0.3675375**	0.545454545	1.833333333	0.545454545	1.833333333
4	3	0.200475	0.204545455	4.888888889	0.204545455	4.888888889
5	4	0.04100625	0	#DIV/0!	0	#DIV/0!
6	5	0	#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!

Sum: 1





INTERPOLATION:
Ratios equal 1 for ...

... x = 1.25 $= (n + 1) / (q/p + 1) - 1$
... r = 2.25 $= (n + 1) / (q/p + 1)$

Round UP to yield maximum probability for ...
... outcome, x = 2 successes.
... term, r = 3 in the series/expansion.

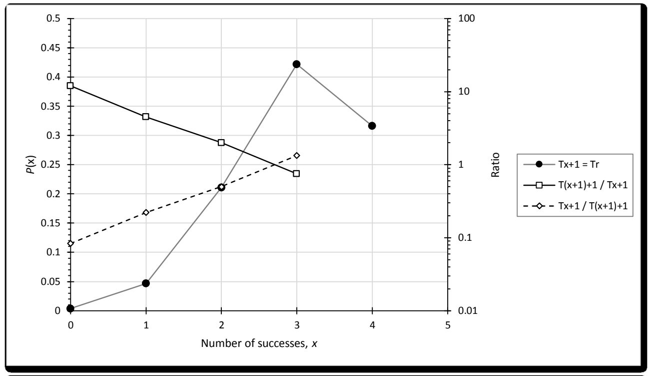
Just for comparison, the 'expected value' = $p\ n$ = 1.8 successes
This neighbours the most probable outcome(s),
which can be obtained (here) by rounding UP.

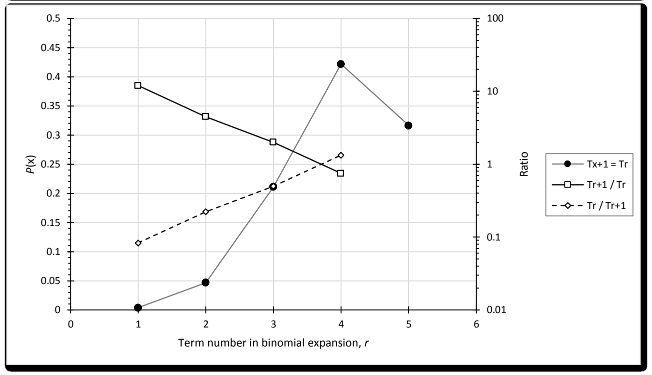
Scenario 5

n = 4
p = 0.75
∴ q = 0.25

Term number	Number of successes	P(x)	Ratios in x:		Ratios in r:	
r	x	$T_{x+1} = T_r$	$T_{(x+1)+1} / T_{x+1}$	$T_{x+1} / T_{(x+1)+1}$	T_{r+1} / T_r	T_r / T_{r+1}
0	-1	0	#DIV/0!	0	#DIV/0!	0
1	0	0.00390625	12	0.083333333	12	0.083333333
2	1	0.046875	4.5	0.222222222	4.5	0.222222222
3	2	0.2109375	2	0.5	2	0.5
4	**3**	**0.421875**	0.75	1.333333333	0.75	1.333333333
5	4	0.31640625	0	#DIV/0!	0	#DIV/0!
6	5	0	#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!

Sum: 1





INTERPOLATION:
Ratios equal 1 for ...

... x = 2.75 $= (n + 1) / (q/p + 1) - 1$
... r = 3.75 $= (n + 1) / (q/p + 1)$

Round UP to yield maximum probability for ...
... outcome, x = 3 successes.
... term, r = 4 in the series/expansion.

Just for comparison, the 'expected value' = $p\ n$ = 3 successes
This neighbours the most probable outcome(s),
which can be obtained (here) DIRECTLY.

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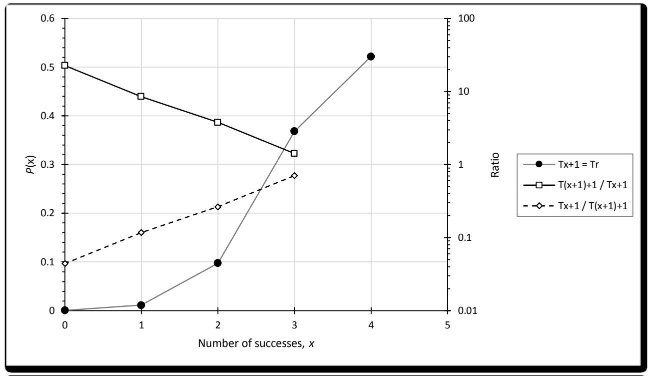
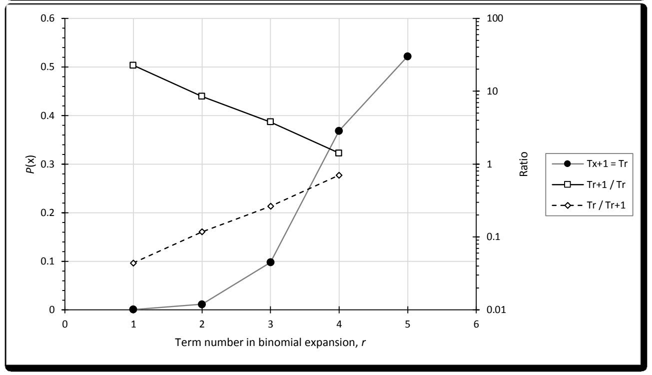
Scenario 6

$n = 4$
$p = 0.85$
$\therefore q = 0.15$

Term number	Number of successes	P(x)	Ratios in x:		Ratios in r:	
r	x	$T_{x+1} = T_r$	$T_{(x+1)+1} / T_{x+1}$	$T_{x+1} / T_{(x+1)+1}$	T_{r+1} / T_r	T_r / T_{r+1}
0	-1	0	#DIV/0!	0	#DIV/0!	0
1	0	0.00050625	22.66666667	0.044117647	22.66666667	0.044117647
2	1	0.011475	8.5	0.117647059	8.5	0.117647059
3	2	0.0975375	3.777777778	0.264705882	3.777777778	0.264705882
4	3	0.368475	1.416666667	0.705882353	1.416666667	0.705882353
5	**4**	**0.52200625**	0	#DIV/0!	0	#DIV/0!
6	5	0	#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!

Sum: 1





EXTRAPOLATION:
Ratios equal 1 for ...

... x = 3.25 $= (n+1) / (q/p + 1) - 1$
... r = 4.25 $= (n+1) / (q/p + 1)$

Round UP to yield maximum probability for ...
... outcome, x = 4 successes.
... term, r = 5 in the series/expansion.

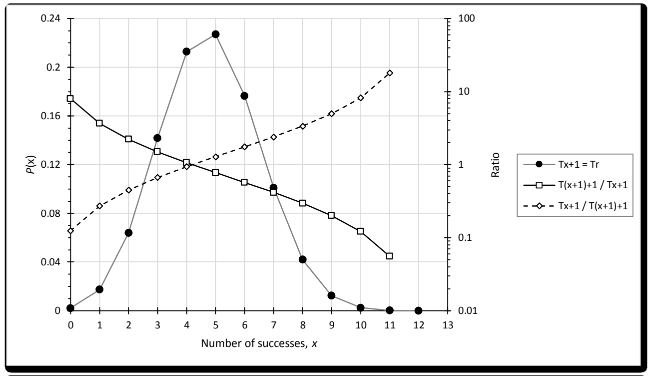
Just for comparison, the 'expected value' = $p\ n$ = 3.4 successes
This neighbours the most probable outcome(s),
which can be obtained (here) by rounding UP.

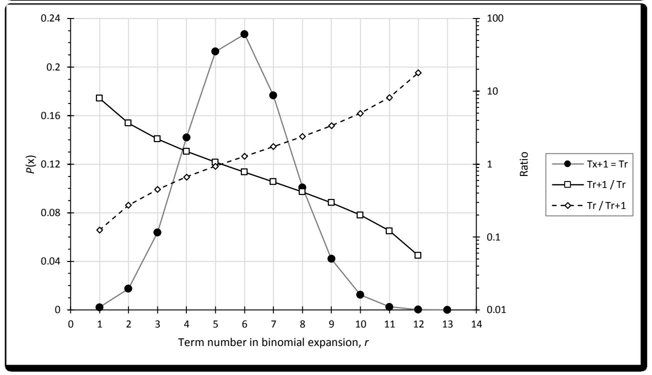
Scenario 7

$n = 12$
$p = 0.4$
$\therefore q = 0.6$

Term number	Number of successes	P(x)	Ratios in x:		Ratios in r:	
r	x	$T_{x+1} = T_r$	$T_{(x+1)+1} / T_{x+1}$	$T_{x+1} / T_{(x+1)+1}$	T_{r+1} / T_r	T_r / T_{r+1}
0	-1	0	#DIV/0!	0	#DIV/0!	0
1	0	0.002176782	8	0.125	8	0.125
2	1	0.017414259	3.666666667	0.272727273	3.666666667	0.272727273
3	2	0.063852282	2.222222222	0.45	2.222222222	0.45
4	3	0.14189396	1.5	0.666666667	1.5	0.666666667
5	4	0.21284094	1.066666667	0.9375	1.066666667	0.9375
6	**5**	**0.227030335**	0.777777778	1.285714286	0.777777778	1.285714286
7	6	0.17657915	0.571428571	1.75	0.571428571	1.75
8	7	0.100902371	0.416666667	2.4	0.416666667	2.4
9	8	0.042042655	0.296296296	3.375	0.296296296	3.375
10	9	0.012457083	0.2	5	0.2	5
11	10	0.002491417	0.121212121	8.25	0.121212121	8.25
12	11	0.00030199	0.055555556	18	0.055555556	18
13	12	1.67772E-05	0	#DIV/0!	0	#DIV/0!
14	13	0	#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!

Sum: 1





INTERPOLATION:
Ratios equal 1 for ...

... x = 4.2 $= (n+1) / (q/p + 1) - 1$
... r = 5.2 $= (n+1) / (q/p + 1)$

Round UP to yield maximum probability for ...
... outcome, x = 5 successes.
... term, r = 6 in the series/expansion.

Just for comparison, the 'expected value' = $p\ n$ = 4.8 successes
This neighbours the most probable outcome(s),
which can be obtained (here) by rounding UP.

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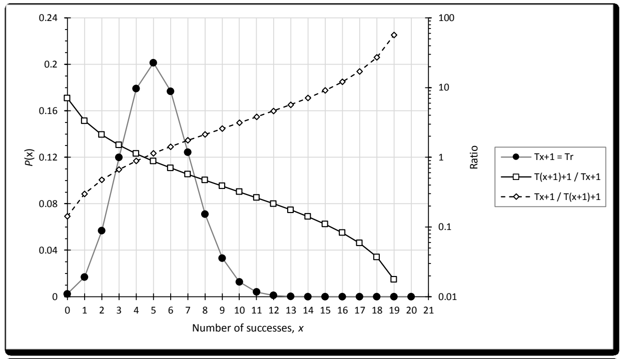
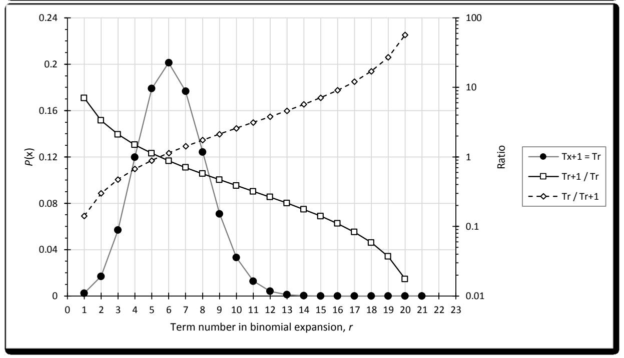
Scenario 8

n = 20
p = 0.26
∴ q = 0.74

Term number	Number of successes	P(x)	Ratios in x:		Ratios in r:	
r	x	$T_{x+1} = T_r$	$T_{(x+1)+1} / T_{x+1}$	$T_{x+1} / T_{(x+1)+1}$	T_{r+1} / T_r	T_r / T_{r+1}
0	-1	0	#DIV/0!	0	#DIV/0!	0
1	0	0.002424568	7.027027027	0.142307692	7.027027027	0.142307692
2	1	0.017037506	3.337837838	0.299595142	3.337837838	0.299595142
3	2	0.056868432	2.108108108	0.474358974	2.108108108	0.474358974
4	3	0.119884802	1.493243243	0.669683258	1.493243243	0.669683258
5	4	0.179017171	1.124324324	0.889423077	1.124324324	0.889423077
6	**5**	**0.201273359**	0.878378378	1.138461538	0.878378378	1.138461538
7	6	0.176794167	0.702702703	1.423076923	0.702702703	1.423076923
8	7	0.124233739	0.570945946	1.75147929	0.570945946	1.75147929
9	8	0.07093075	0.468468468	2.134615385	0.468468468	2.134615385
10	9	0.03322882	0.386486486	2.587412587	0.386486486	2.587412587
11	10	0.01284249	0.319410319	3.130769231	0.319410319	3.130769231
12	11	0.004102024	0.263513514	3.794871795	0.263513514	3.794871795
13	12	0.001080939	0.216216216	4.625	0.216216216	4.625
14	13	0.000233716	0.175675676	5.692307692	0.175675676	5.692307692
15	14	4.10583E-05	0.140540541	7.115384615	0.140540541	7.115384615
16	15	5.77036E-06	0.109797297	9.107692308	0.109797297	9.107692308
17	16	6.33569E-07	0.082670906	12.09615385	0.082670906	12.09615385
18	17	5.23778E-08	0.058558559	17.07692308	0.058558559	17.07692308
19	18	3.06717E-09	0.036984353	27.03846154	0.036984353	27.03846154
20	19	1.13437E-10	0.017567568	56.92307692	0.017567568	56.92307692
21	20	1.99281E-12	0	#DIV/0!	0	#DIV/0!
22	21	0	#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!

Sum: 1





INTERPOLATION:
Ratios equal 1 for ...

... x = 4.46 = $(n + 1) / (q/p + 1) - 1$
... r = 5.46 = $(n + 1) / (q/p + 1)$

Round UP to yield maximum probability for ...
... outcome, x = 5 successes.
... term, r = 6 in the series/expansion.

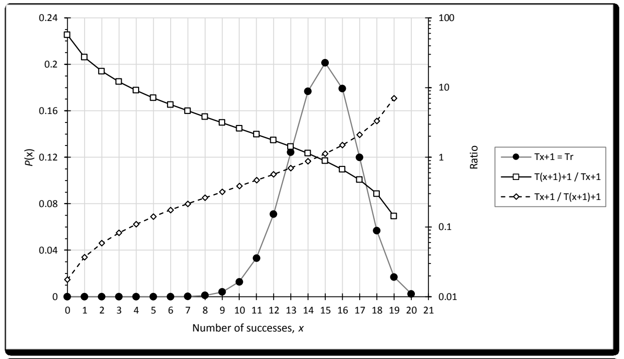
Just for comparison, the 'expected value' = $p\ n$ = 5.2 successes
This neighbours the most probable outcome(s),
which can be obtained (here) by rounding DOWN.

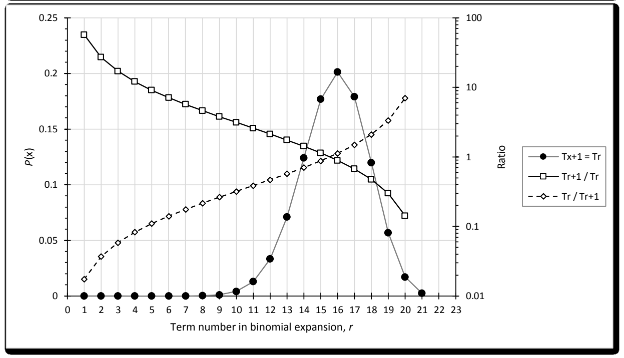
Scenario 9

n = 20
p = 0.74
∴ q = 0.26

Term number	Number of successes	P(x)	Ratios in x:		Ratios in r:	
r	x	$T_{x+1} = T_r$	$T_{(x+1)+1} / T_{x+1}$	$T_{x+1} / T_{(x+1)+1}$	T_{r+1} / T_r	T_r / T_{r+1}
0	-1	0	#DIV/0!	0	#DIV/0!	0
1	0	1.99281E-12	56.92307692	0.017567568	56.92307692	0.017567568
2	1	1.13437E-10	27.03846154	0.036984353	27.03846154	0.036984353
3	2	3.06717E-09	17.07692308	0.058558559	17.07692308	0.058558559
4	3	5.23778E-08	12.09615385	0.082670906	12.09615385	0.082670906
5	4	6.33569E-07	9.107692308	0.109797297	9.107692308	0.109797297
6	5	5.77036E-06	7.115384615	0.140540541	7.115384615	0.140540541
7	6	4.10583E-05	5.692307692	0.175675676	5.692307692	0.175675676
8	7	0.000233716	4.625	0.216216216	4.625	0.216216216
9	8	0.001080939	3.794871795	0.263513514	3.794871795	0.263513514
10	9	0.004102024	3.130769231	0.319410319	3.130769231	0.319410319
11	10	0.01284249	2.587412587	0.386486486	2.587412587	0.386486486
12	11	0.03322882	2.134615385	0.468468468	2.134615385	0.468468468
13	12	0.07093075	1.75147929	0.570945946	1.75147929	0.570945946
14	13	0.124233739	1.423076923	0.702702703	1.423076923	0.702702703
15	14	0.176794167	1.138461538	0.878378378	1.138461538	0.878378378
16	**15**	**0.201273359**	0.889423077	1.124324324	0.889423077	1.124324324
17	16	0.179017171	0.669683258	1.493243243	0.669683258	1.493243243
18	17	0.119884802	0.474358974	2.108108108	0.474358974	2.108108108
19	18	0.056868432	0.299595142	3.337837838	0.299595142	3.337837838
20	19	0.017037506	0.142307692	7.027027027	0.142307692	7.027027027
21	20	0.002424568	0	#DIV/0!	0	#DIV/0!
22	21	0	#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!

Sum: 1





INTERPOLATION:
Ratios equal 1 for ...

... x = 14.54 = $(n + 1) / (q/p + 1) - 1$
... r = 15.54 = $(n + 1) / (q/p + 1)$

Round UP to yield maximum probability for ...
... outcome, x = 15 successes.
... term, r = 16 in the series/expansion.

Just for comparison, the 'expected value' = $p\ n$ = 14.8 successes
This neighbours the most probable outcome(s),
which can be obtained (here) by rounding UP.